Radian Measure, Arc Length, and Area

Unit circle: A circle of radius one that is centered at the origin.

What is the circumference of the unit circle? (Circumference of a circle: $C = \pi d = 2\pi r$)

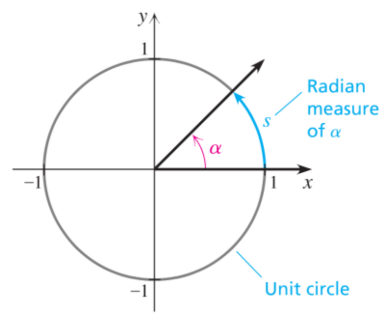
What is the arc length intercepted by a 180° angle (1/2 of the circle)?

What is the arc length intercepted by a 120° angle (1/3 of the circle)?

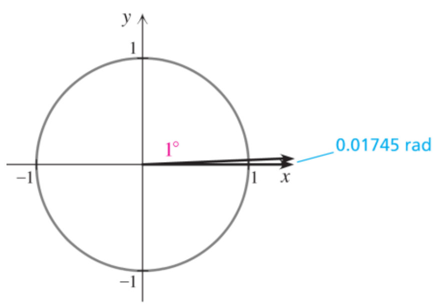
What is the arc length intercepted by a 30° angle? An angle of 225°? An angle of 210°?

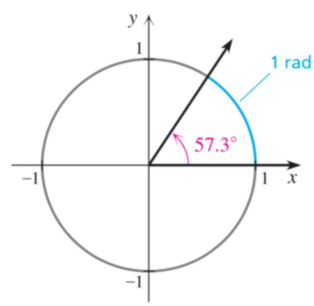
You have just calculated the radian measure of each of these angles.

The **radian measure** of the angle α in standard position is the directed length of the intercepted arc on the unit circle. **Directed length** means that the radian measure is positive if the rotation of the terminal side is counterclockwise and negative if the rotation is clockwise.



One radian: The angle that intercepts an arc with length equal to the radius of a circle. (On the unit circle, one radian is the angle that intercepts an arc of length one.)





Since the radius of the unit circle is the real number 1, without any dimension (ft., meters, etc.), the arc length, and hence the radian measure of an angle, is also a real number without any dimension. Thus, one radian (1 rad) is the real number 1.

Converting Between Radians and Degrees:

Since there are 2π radians in a circle (the circumference of the unit circle is 2π) and 360° in a circle,

$$2\pi \text{ radians} = 360°, \text{ or } \pi \text{ radians} = 180°.$$

Degrees→Radians: multiply by $\dfrac{\pi \text{ rad}}{180°}$. **Radians→Degrees:** multiply by $\dfrac{180°}{\pi \text{ rad}}$.

Examples:

Convert the degree measures to radians:

a) 210°　　　　b) –27.2°

Convert the radian measures to degrees:

a) $\frac{5\pi}{3}$　　　　b) 16.7 radians

Examples: Draw each angle in standard position and determine the quadrant in which each angle lies.

a) $-\frac{5\pi}{4}$　　　　b) $\frac{10\pi}{3}$　　　　c) 13.8　　　　d) -2.5

Coterminal Angles:

To find coterminal angles in radians, add or subtract multiples of 2π. Make sure to find a common denominator.

Examples: Find two positive angles and two negative angles that are coterminal with the given angle.

a) $\frac{5\pi}{6}$　　　　b) $-\frac{\pi}{4}$　　　　c) $\frac{7\pi}{3}$　　　　d) 1.4

Arc Length

Often, we want to find the arc length (s) on a circle of radius r, intercepted by an angle α. We can do this by determining what fraction of the circle we are looking at, then multiplying by the circumference of the circle.

Degrees: $s = \frac{\alpha}{360°} \cdot 2\pi r$

Radians: $s = \frac{\alpha}{2\pi} \cdot 2\pi r$　　or　　$\boldsymbol{s = \alpha r}$　　　**This formula only works if $\boldsymbol{\alpha}$ is in radians!**

Examples:

A central angle of $\pi/2$ intercepts an arc on the surface of the earth that runs from the equator to the North Pole. Using 3950 miles as the radius of the earth, find the length of the intercepted arc to the nearest mile.

A wagon wheel has a diameter of 28 inches and an angle of 30° between the spokes. What is the length of the arc *s* (to the nearest hundredth of an inch) between two adjacent spokes?

Area of a Sector of a Circle

Finding the area (*A*) of a sector of a circle of radius *r* with central angle *α*, is similar to finding the arc length: Determine what fraction of the circle the sector makes up, then multiply by the area of the circle.

Degrees: $A = \frac{\alpha}{360°} \cdot \pi r^2$

Radians: $A = \frac{\alpha}{2\pi} \cdot \pi r^2$ or $\boldsymbol{A = \frac{\alpha r^2}{2}}$ **This formula only works if *α* is in radians!**

Examples:

Which is bigger: a slice of pizza from a 10″ diameter pizza cut into 6 slices, or a slice from a 12″ diameter pizza cut into 8 slices?

A center-pivot irrigation system is used to water a circular field with radius 200 feet. In three hours the system waters a sector with a central angle of $\pi/8$. What area (in square feet) is watered in that time?